Math 579 Fall 2013 Exam 2 Solutions

1. Prove that $\sum_{i=1}^{n} i(i-3) = \frac{(n-4)n(n+1)}{3}$.

   Proof by induction on n, natch. Base case: $n=1$, LHS$=1(1-3)=-2=\frac{(1-4)1(1+1)}{3}$=RHS. Assume that $\sum_{i=1}^{n} i(i-3) = \frac{(n-4)n(n+1)}{3}$ holds, and add $(n+1)(n+1-3)$ to both sides. Then $\sum_{i=1}^{n+1} i(i-3) = \frac{(n-4)n(n+1)}{3} + (n+1)(n-2) = (n+1)(\frac{n^2-4n}{3} + \frac{3n-6}{3}) = (n+1)(\frac{n^2-n-6}{3}) = (n+1)\frac{(n-3)(n+2)}{3} = \frac{(n+1-4)(n+1)(n+1+1)}{3}$, as desired.

2. Let $a_1 = 1, a_2 = 5$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$. Prove that $a_n = 2^n + (-1)^n$.

   Proof by strong induction on n. Two base cases: $n = 1$ $a_1 = 1 = 2^1 + (-1)^1$ and $a_2 = 5 = 2^2 + (-1)^2$.
   Now we have $a_n = a_{n-1} + 2a_{n-2}$. Applying the inductive hypothesis to each summand, we get $a_n = 2^{n-1} + (-1)^{n-1} + 2(2^{n-2} + (-1)^{n-2}) = (2^{n-1} + 2 \cdot 2^{n-2}) + (-1)^{n-2}(-1+2) = 2^n + (-1)^n$, as desired.

3. Recall that F_i denotes the Fibonacci numbers, i.e. $F_1 = F_2 = 1$ and $F_j + F_{j+1} = F_{j+2}$ for $j \geq 1$. Prove that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$, for all $n \in \mathbb{N}$.

   Proof by induction on n. Base case: $n = 1$, LHS$=F_1^2 = 1 = F_1 F_2$.
   Assume now that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$. Add F_{n+1}^2 to both sides, that gives $\sum_{i=1}^{n+1} F_i^2 = F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) = F_{n+1} F_{n+2}$, as desired.

4. Use induction to prove that $\frac{d}{dx} x^n = n x^{n-1}$, for all $n \in \mathbb{N}$.

   Proof by induction on n. Base case $n = 1$, we use the definition of derivative to find $\frac{d}{dx} x^1 = \lim_{\epsilon \to 0} \frac{(x+\epsilon)-x}{\epsilon} = \lim_{\epsilon \to 0} 1 = 1$.
   We now use the product rule as $\frac{d}{dx}(x^n) = \frac{d}{dx}(x \cdot x^{n-1}) = 1 \cdot x^{n-1} + x \cdot (n-1)x^{n-2} = x^{n-1}(1+n-1) = x^{n-1}(n)$, where we used the inductive hypothesis to conclude that $\frac{d}{dx} x^{n-1} = (n-1)x^{n-2}$.

5. A *tree* is a connected simple finite graph with no cycles. Prove that every tree on n vertices must have exactly $n-1$ edges. You may use freely the following result:
   Theorem: Every tree with at least two vertices has at least two leaves.

   Proof by induction on n. Pre-base-case: If $n = 1$, then the tree has no edges, so the result is true. Base case: If $n = 2$, then since the tree is connected the two vertices must have an edge between them, so there is exactly one edge, so the result is true.
   Inductive case: Consider an arbitrary tree on n vertices, and choose a leaf vertex whose existence is guaranteed by the theorem. Deleting this vertex and its lone attached edge gives a smaller graph. The resulting graph is still connected, simple, finite, and has no cycles, hence is a tree with $n-1$ vertices. By the inductive hypothesis, it has $n-2$ edges. Thus, the original tree must have had $n-1$ edges.